# Nonlocal effects in high-energy charged-particle beams

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(Received 11 November 2003; published 2 June 2004)

Within the framework of the thermal wave model, an investigation is made of the longitudinal dynamics of high-energy charged-particle beams. The model includes the nonlinear self-consistent interaction between the beam and its surroundings in terms of a coupling impedance, and when resistive as well as reactive parts are included, the evolution equation becomes a generalized nonlinear Schrödinger equation including a nonlocal nonlinear term. The consequences of the resistive part on the propagation of particle bunches are examined using analytical as well as numerical methods.

DOI: 10.1103/PhysRevE.69.066501 PACS number(s): 41.85.−p, 03.65.Ca, 42.50.−p

## I. INTRODUCTION

The thermal wave model (TWM) [1–3] describes the dynamics of high-energy charged-particle beams in accelerators. In the TWM approach, the beam is characterized by a complex-valued wave function, which satisfies a Schrödinger-like evolution equation, where the intensity of the wave function corresponds to the beam particle density. The Schrödinger potential, which describes the interaction between the beam and its surroundings, can be expressed in terms of a coupling impedance, and due to collective effects, it is a nonlinear function of the beam density. For purely reactive impedances, the TWM equation reduces to the well-known nonlinear Schrödinger equation. However, by including the resistive part, the evolution equation becomes a generalized Schrödinger equation containing a new term, which is both nonlinear and nonlocal. The modulational instability properties of this new equation have been analyzed previously [4] and have been shown to agree with results obtained using classical approaches, including kinetic effects such as Landau damping [5].

In the present work we consider the dynamics of particle bunches under the influence of the coupling impedance. Since the purely reactive case is well known, main emphasis is on the situation where the resistive part is included. The dynamical evolution then proceeds as a competition between linear dispersion, nonlinear self-focusing/defocusing, and nonlocal self-steepening. It is found that the bunch is accelerated/decelerated, and the self-steepening effect makes the pulse shape asymmetric with an extended tail, and eventually a wave-breaking-like phenomenon can appear on the steepening edge.

The paper is organized as follows. In Sec. II, the basic model equation is described, and some limiting cases are discussed. In Sec. III, which is based on a variational approach, the evolution of a perturbed soliton solution in the presence of a small resistive coupling impedance is presented. In Sec. IV, the nonlocal effects are described using a direct perturbation analysis. These results illustrate the interplay between the different effects and are qualitatively in agreement with results obtained by other means, as summarized, e.g., in Ref. [6]. The wave-breaking phenomenon is predicted in Sec. V, and a corresponding characteristic length-scale is found. Stationary solutions are considered in Sec. VI, showing the existence of semi-infinite shock solutions and the nonexistence of pulselike solutions with a finite particle number. All analytical predictions are confirmed by numerical simulations of the full model equation. Finally, in Sec. VII the conclusions are summarized.

## II. THE GENERALIZED NONLINEAR SCHRÖDINGER EQUATION

Within the TWM, the longitudinal dynamics of particle bunches are analyzed in terms of a complex beam wave function $\Psi(\xi,\zeta)$, where $\zeta$ is the distance of propagation and $\xi$ is the longitudinal extension of the particle beam, measured in the moving frame of reference. The particle density $\lambda(\xi,\zeta)$ is related to the wave function according to $\lambda(\xi,\zeta) = |\Psi(\xi,\zeta)|^2$ [1]. The collective longitudinal evolution of the beam in a circular high-energy accelerating machine is described by the Schrödinger-like equation

$$i\epsilon_x \frac{\partial \Psi}{\partial \zeta} = \frac{\epsilon_x^2 \eta}{2}\frac{\partial^2 \Psi}{\partial \xi^2} + U(\xi,\zeta)\Psi, \tag{1}$$

where $\epsilon_x$ is the longitudinal beam emittance and $\eta$ is the slip factor [7] defined as $\eta = \gamma_T^{-2} - \gamma^{-2}$ ($\gamma_T$ being the transition energy, defined as the inverse of the momentum compaction [7] and $\gamma$ being the relativistic factor); $U(\xi,\zeta)$ is the effective

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dimensionless (with respect to the nominal particle energy, $E_0 = m\gamma c^2$) potential energy given by the interaction between the bunch and the surroundings. Note that $\eta$ can be positive (above transition energy) or negative (below transition energy). Above transition energy, in analogy with quantum mechanics, $1/\eta$ plays the role of an effective mass associated with the beam as a whole. Below transition energy, $1/\eta$ plays the role of a "negative mass"; in analogy with optics, this physical circumstance corresponds to anomalous dispersion.

Equation (1) has to be coupled with an equation for $U$. If no external sources of electromagnetic (EM) fields are considered and the effects of charged-particle radiation damping is negligible, the (self-)interaction of the beam with the surroundings is determined by the image charges and the image currents appearing on the walls of the vacuum chamber. We consider a torus-shaped accelerating machine, characterized by a toroidal radius $R_0$ and a poloidal radius $b$. The self-interaction is suitably described in terms of the "longitudinal coupling impedance" [7] whose real and imaginary parts account for the resistive and the total reactive (capacitive and inductive) effects, respectively. By calculating the EM fields and the induced wall currents due to deviations from the nominal distribution, the following equation for the self-force acting back on the system can be established [see Eq. (10) in Ref. [8]]

$$(1-L)\epsilon'' + R\epsilon' - \sigma_1\epsilon = \sigma_2[R\lambda_1 + (g_0 - L)\lambda_1'], \tag{2}$$

where the derivatives are taken with respect to the normalized longitudinal coordinate, and $\epsilon$ and $\lambda_1$ are the normalized self-field and density perturbation, respectively; $R$ and $L$ are the dimensionless resistance and inductance, respectively; and $g_0\lambda_1'$ accounts for the capacitive space-charge effect, $g_0$ being a geometry factor (for all the details, see Ref. [8]). Here, the dimensionless parameters $\sigma_1$ and $\sigma_2$ are defined as $\sigma_1 = (4\pi\gamma R_0/b)^2$ and $\sigma_2 = (2\eta N R_0/\epsilon_0 E_0)(q/\beta b)^2$, respectively, where $N$ is the beam particle number, $q$ the particle charge, $\epsilon_0$ the vacuum dielectric constant, and $\beta$ is the particle speed in the unit $c$ (speed of light). Once Eq. (2) is solved for $\epsilon$, the potential is obtained by space integration. This means that the effective potential energy $U$ to be inserted in Eq. (1) is a functional of the density perturbation $\lambda_1$. In order to find the explicit form of this functional we observe that $\sigma_1 \gg 1$. Furthermore, we confine our analysis to phenomena that do not fall into the category of strongly nonlinear problems, i.e., the wave amplitude cannot be extrapolated to arbitrarily high values; additionally, the gradients of the physical quantities (density, electric field, etc.) are not very large. Consequently, under the above physical assumptions, the first two terms in the left-hand side of Eq. (2) can be neglected. Thus, the integration with respect to $\xi$ of the resulting longitudinal electric field allows us to obtain the functional form for $U$ as

$$U[\lambda_1(\xi,\zeta)] = \frac{q^2\beta c}{E_0}\left(R_0 Z_I' \lambda_1(\xi,\zeta) + Z_R' \int_0^{\xi} \lambda_1(\xi',\zeta)\,d\xi'\right), \tag{3}$$

where $Z_R'$ and $Z_I'$ are the resistive and the total reactive parts, respectively, of the longitudinal coupling impedance per unit length of the machine. Thus, the coupling impedance per unit length can be defined as the complex quantity $Z = Z_R + iZ_I$. In our simple model of a circular machine, it is easy to see that

$$Z_I' = \frac{1}{2\pi R_0}\left(\frac{g_0 Z_0}{2\beta\gamma^2} - \omega_0\mathcal{L}\right) \equiv \frac{Z_I}{2\pi R_0}, \tag{4}$$

where $Z_0$ is the vacuum impedance, $\omega_0 = \beta c/R_0$ is the nominal orbital angular frequency of the particles, and $\mathcal{L}$ is the total inductance. This way, $Z_I$ represents the total reactance as the difference between the total space charge capacitive reactance $g_0 Z_0/(2\beta\gamma^2)$ and the total inductive reactance $\omega_0\mathcal{L}$. Consequently, in the limit of negligible resistance, Eq. (3) reduces to

$$U[\lambda_1] = \frac{q^2\beta c}{2\pi E_0}\left(\frac{g_0 Z_0}{2\beta\gamma^2} - \omega_0\mathcal{L}\right)\lambda_1. \tag{5}$$

Provided that the first two terms in the left-hand side of Eq. (2) are negligible, Eq. (5) coincides with the usual expression of the effective potential energy used in the case of a purely reactive impedance [7]. Actually, Eq. (3) then fully agrees with the effective potential energy given in the literature, although the derivation is given there in terms of electric circuit models or within the theory of wake fields [9,10].

Under certain physical conditions, the two "extra terms" in Eq. (2) play an important role. In the classical kinetic theory of a coasting beam, the system, described by Eq. (2) and the Vlasov equation, predicts some nonlinear effects that are not present in the simplified description [8]. In fact, the existence of some nonlinear coherent localized structures, such as solitons, holes, etc., that have been numerically as well as experimentally observed [11–13] requires the nonlinearity due to the large amplitude regime to be balanced by sufficiently strong dispersion. This effect is not present in the standard kinetic description of coasting beam dynamics without the extra terms. Recent theoretical investigations carried out within the classical Vlasov theory, based on the full Eq. (2), have predicted such coherent localized structures [8,14,15] in good agreement with both numerical and experimental observations.

However, as mentioned, our present analysis is carried out in the regime where the two extra terms can be neglected and, additionally, the model we are using is not the Vlasov one. In the framework of the TWM, we have to express Eq. (3) in terms of the beam wave function and then substitute in Eq. (1). Let us denote the initial beam density by $\lambda(\xi,0)$. Since we are assuming that the unperturbed beam is coasting, we can put $\lambda(\xi,0) = \lambda_0$, where $\lambda_0$ is a positive constant (initially, the beam is uniformly distributed along $\xi$ with density $\lambda_0$). According to the TWM assumptions described in Sec. I, we have $\lambda_0 = |\Psi(\xi,0)|^2 \equiv |\Psi_0|^2$, where $\Psi_0$ is a complex function, whose squared modulus is the initial unperturbed density of the coasting beam. On the basis of the above TWM interpretations, we can write the density perturbation as $\lambda_1(\xi,\zeta) = |\Psi(\xi,\zeta)|^2 - |\Psi_0|^2$. Thus, the combination of Eq. (1) and Eq. (3) gives the following evolution equation for the beam:

$$i\frac{\partial \Psi}{\partial \zeta} = \alpha\frac{\partial^2\Psi}{\partial \xi^2} + \kappa(|\Psi|^2 - |\Psi_0|^2)\Psi + \mu\Psi\int_0^{\xi}[|\Psi(\xi',\zeta)|^2 - |\Psi_0|^2]d\xi', \qquad (6)$$

where

$$\alpha = \eta/2 = (\gamma_T^{-2} - \gamma^{-2})/2, \qquad (7)$$

$$\kappa = q^2\beta c R_0/(\epsilon_x E_0)Z_I', \qquad (8)$$

$$\mu = q^2\beta c/(\epsilon_x E_0)Z_R'. \qquad (9)$$

Equation (6) has recently been used to study the Landau-type damping of large amplitude EM wave packets in a nonlinear medium as well as relatively intense high-energy charged-particle coasting beams in accelerating machines [5,16].

The fundamental evolution equation of the TWM, Eq. (6), can be written in several different, but equivalent, ways. For the purpose of the present investigation, it is convenient to use a somewhat simpler form by making the transformation

$$\Psi(\xi,\zeta) = \psi(x,z)\exp[i\Theta(x,z)], \qquad (10)$$

where

$$\Theta(x,z) = \mu|\Psi_0|^2 xz - \tfrac{2}{3}\alpha\mu^2|\Psi_0|^4 z^3 + \kappa|\Psi_0|^2 z, \qquad (11)$$

and the new coordinates are defined by

$$x = \xi + \alpha\mu|\Psi_0|^2\zeta^2, \qquad (12)$$

$$z = \zeta. \qquad (13)$$

Furthermore, the lower limit of integration in Eq. (6) can be extended to minus infinity, the difference being only a $z$-dependent phase variation of the beam wave function. The resulting form of the thermal wave model equation, to be used in the present work, then reads

$$i\frac{\partial \psi}{\partial z} = \alpha\frac{\partial^2\psi}{\partial x^2} + \kappa|\psi|^2\psi + \mu\psi\int_{-\infty}^{x}|\psi(x',z)|^2dx'. \qquad (14)$$

In the case $\mu=0$, Eq. (14) reduces to the fundamental nonlinear Schrödinger equation for which a wealth of information is available. In particular, depending on the sign of the product $\alpha\kappa$, the nonlinearity will either counteract $(\alpha\kappa>0)$ or enhance $(\alpha\kappa<0)$ the dispersive broadening. Furthermore, the velocity of the particle bunch is left unchanged, and no asymmetry is introduced on an initially symmetric bunch. Of special interest is the case $\alpha\kappa>0$, when shape-preserving soliton solutions are possible as a balance between linear dispersion and nonlinear self-focusing effects.

The properties of the full Eq. (14) are not known and in order to see the physical significance of the new term, it is instructive to qualitatively discuss the nonlinear potential $U[\psi]$, which in the case $\alpha<0$ is given by

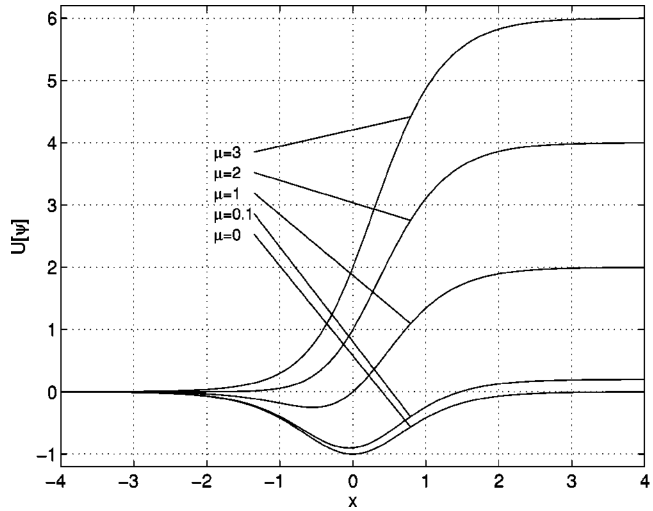


FIG. 1. A qualitative plot of the potential, Eq. (15), for a pulse-shaped field using different values of $\mu$.

$$U[\psi] = \kappa|\psi|^2 + \mu\int_{-\infty}^{x}|\psi|^2dx'. \qquad (15)$$

Consider first the case $\mu\rightarrow 0$ and $\alpha\kappa>0$, i.e., assume that also $\kappa<0$. Then, sech-shaped solutions form a well-shaped potential that allows bound states, solitons, to exist. The fundamental soliton solution corresponding to Eq. (14) is

$$\psi = A_0\,\mathrm{sech}(ax)e^{-i\delta z}, \quad a=\sqrt{\frac{\kappa A_0^2}{2\alpha}}, \quad \delta=\frac{\kappa A_0^2}{2}. \qquad (16)$$

The nonlocal part of the potential introduced by $\mu\neq 0$ contributes a monotonous term to the potential and creates an asymmetry. A qualitative plot of the total potential corresponding to a field shaped as the fundamental soliton (choosing $A_0=1$, $a=1$, and $\kappa=-1$), is shown in Fig. 1, using different values for $\mu$. It is clear that if $\mu$ is small, the evolution of an initially soliton-shaped pulse should involve an acceleration in a direction determined by the sign of $\mu$, but the change of the pulse shape can be expected to be slow due to the similarities with the conditions for soliton propagation. For large $\mu$, however, there can be no pulse-shaped stationary solutions, since the total potential then is monotonous, and thus is unable to provide the confining effects needed for soliton generation. By noticing that the slope of the potential varies over the pulse, and that the intense parts are accelerated/decelerated more than the low intensity parts, strong internal pulse dynamics is anticipated. Our subsequent analysis will confirm this intuitive picture.

### III. PERTURBED SOLITON DYNAMICS

From the potential picture it is expected that one of the main effects of the nonlocal term is to induce an acceleration of an initially stationary pulse. A more quantitative analysis of this effect can be carried out by investigating the adiabatic evolution of the soliton solution, Eq. (16), in the presence of

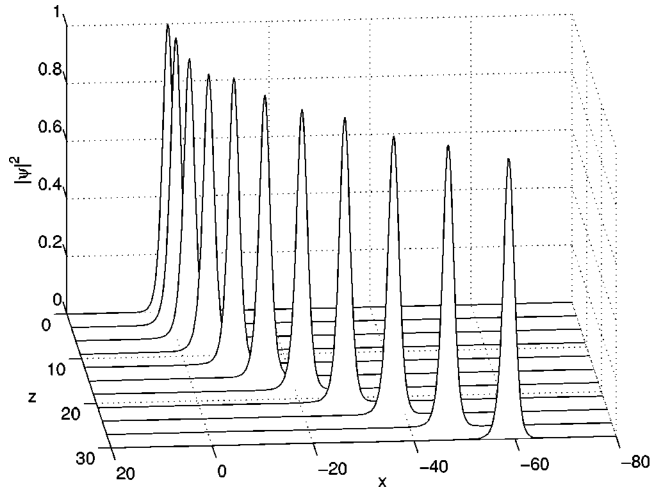


FIG. 2. The numerically obtained dynamics of an initially soliton-shaped pulse using a weak nonlocal term.

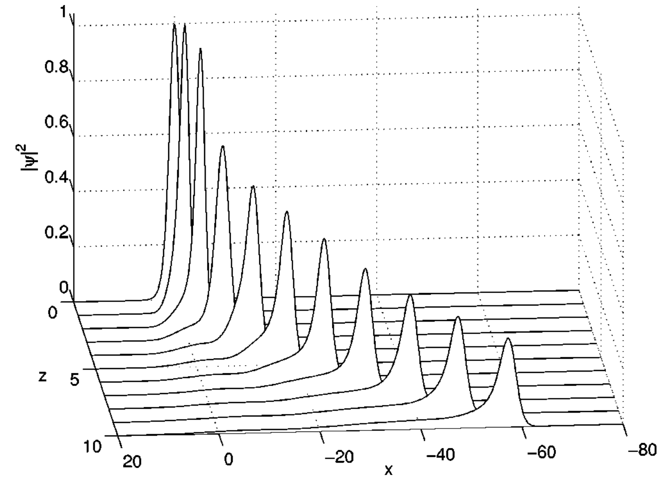


FIG. 3. By increasing the strength of the nonlocal term, strong pulse shape distortion is introduced.

a small but finite value of $\mu$. This can conveniently be done using a direct variational approach, see, e.g., Ref. [17]. A suitable trial function is

$$\psi_T(x,z) = A\ \mathrm{sech}[a(x-M)]e^{i[C(x-M)+D]}, \tag{17}$$

where $A(z)$, $a(z)$, $C(z)$, $D(z)$, and $M(z)$ are unknown parameter functions to be determined by the variational procedure. We emphasize that this ansatz function neglects any asymmetric pulse deformations and consequently can model only part of the dynamical evolution. Using Ritz optimization, these parameter functions can be determined and the following approximate solution is obtained:

$$\psi_T = A_0\ \mathrm{sech}\left(\sqrt{\frac{\kappa A_0^2}{2\alpha}}s\right)\exp\left\{i\left[-\frac{2\mu A_0^2}{3}zs - \frac{4\alpha\mu^2 A_0^4}{27}z^3 - \left(\mu A_0^2\sqrt{\frac{2\alpha}{\kappa A_0^2}} + \frac{\kappa A_0^2}{2}\right)z\right]\right\}, \tag{18}$$

where

$$s = x - \frac{2\alpha\mu A_0^2}{3}z^2. \tag{19}$$

This solution is consistent with the classical nonlinear Schrödinger (NLS) equation in the sense that the fundamental soliton, Eq. (16), is recovered in the limit when $\mu \to 0$. The solution in the general case describes a soliton being accelerated in the original frame of reference. The acceleration $\gamma$ is given by $\gamma = 4\alpha\mu A_0^2/3$ and the concomitant shift of the group velocity is associated with a frequency shift proportional to $z$.

In order to check the approximate analytical solutions, but also to obtain results in parameter ranges where analytical solutions are not available, Eq. (14) has been solved numerically. For this purpose, the standard split-step Fourier method for handling the NLSE is modified to include the effects of the nonlocal term. An example of the dynamics caused by a weak nonlocal term is seen in Fig. 2. The initial pulse, which is centered around $x=0$, is the fundamental soliton, making a comparison with the variational result simple. The parameters are $\alpha=-1$, $\kappa=-1$, $\mu=0.1$, and $A_0=1$, and the propagation distance is $L=30$. The pulse is accelerated towards negative $x$ coordinates, but the shape is only weakly distorted. Thus, for small values of $\mu$, the variational result, which predicts the final center position to be $x=-60$, describes the propagation in an excellent way.

However, for larger values of $\mu$, the asymmetric deformation of the pulse becomes more important and tends to leave an extended tail behind the main pulse. A numerical simulation result of this situation is shown in Fig. 3, where $\mu=1$. The propagation distance is $L=10$, making the final $x$ position similar to the example above. It is seen that the internal dynamics of the pulse is significantly stronger; the peak power decreases, and a wake field is developed. Again using the variational result, the final position is expected to be $x=-66.7$, but the decaying peak power makes the potential less steep during propagation, which causes the acceleration to decrease.

Clearly, situations where $\mu$ is large cannot be analyzed by a variational approach based on the soliton ansatz. In order to study the deformation dynamics in some more analytical detail, we will instead use a perturbation analysis.

## IV. PERTURBATION ANALYSIS

Although a general solution of Eq. (14) based on analytical methods is not possible, the initial dynamics can be described using a perturbation analysis. For this purpose, Eq. (14) is rewritten as a coupled system in the real amplitude $A$ and the phase $\theta$ of $\psi$ according to

$$\frac{\partial A^2}{\partial z} = 2\alpha\frac{\partial}{\partial x}\left(A^2\frac{\partial\theta}{\partial x}\right), \tag{20}$$

$$\frac{\partial\theta}{\partial z} = -\alpha\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^2} - \left(\frac{\partial\theta}{\partial x}\right)^2\right] - \kappa A^2 - \mu\int_{-\infty}^{x} A^2 dx'. \tag{21}$$

For an initially unchirped pulse, i.e., $\theta(x,0)=0$, the initial amplitude modulation first creates a phase modulation proportional to $z$, which then generates a subsequent change, proportional to $z^2$, of the amplitude modulation. Let us consider the case of the fundamental soliton, Eq. (16), as initial field, since the dispersion and the Kerr nonlinearity then balance each other. Thus, as initial condition we consider $A(x,z=0)=A_0\ \mathrm{sech}(ax)$, where $a$ is related to $A_0$ according to Eq. (16), and $\theta(x,z=0)=0$. Using these in the right-hand side of Eq. (21), the initial evolution of $\theta$ is obtained, and this solution can then be used to find the lowest order modifications of $A$ from Eq. (20).

However, as found using variational analysis, the pulse evolution is, due to the effects of the nonlocal term, most conveniently described in an accelerated coordinate system. The proper value of the acceleration can be taken from the preceding section, but it is also instructive to derive it using an analogy with Ehrenfest's theorem in quantum mechanics. It is straightforward to show that the motion of the mean position $\langle x\rangle$ of the bunch obeys the equation of motion

$$\gamma_0 \equiv \frac{d^2\langle x\rangle}{dz^2} = -2\alpha\langle F\rangle, \tag{22}$$

where the averaging is defined according to

$$\langle f\rangle \equiv \frac{\int_{-\infty}^{\infty} f|\psi|^2 dx}{\int_{-\infty}^{\infty} |\psi|^2 dx}, \tag{23}$$

and the force is $F=-\partial U/\partial x$. The acceleration obtained in this way is identical to that derived using the variational approach. Thus, $\gamma_0=\gamma$ and the new coordinate $s$ is defined according to Eq. (19). The amplitude and phase are then obtained as

$$A = A_0\ \mathrm{sech}(as)\sqrt{1+4\alpha a\mu A_0^2 \tanh(as)\left[\mathrm{sech}^2(as)-\tfrac{1}{3}\right]z^2}, \tag{24}$$

$$\theta = -\left\{\frac{\kappa A_0^2}{2} + \frac{\mu A_0^2}{a}[\tanh(as)+1]\right\}z. \tag{25}$$

The first part of the phase, Eq. (25), does not depend on $s$, and is identical to the phase of the fundamental soliton. The second part is tanh shaped, which is due to the form of the nonlocal potential term. As expected from the potential picture, it is found that the amplitude becomes asymmetric.

Due to the limited accuracy, the perturbation analysis can only be applied within a certain propagation distance, obtained from Eq. (24) as

$$4\alpha a\mu A_0^2 z^2 \ll 1 \Rightarrow z \ll 1/\sqrt{|4\alpha a\mu A_0^2|}. \tag{26}$$

Using the same numerical parameters as above, no significant changes in the amplitude are seen within that range. However, as seen in Eq. (26), the application range decreases slowly as $\mu$ increases. Thus, a large value, $\mu=10$, has been used in Fig. 4, where the perturbation analysis is compared with the numerically obtained result after a propagation distance $L=0.2$. It is seen that the pulse is starting to "lean to the side," and that the perturbation profile is in good agreement with the numerical result, although its peak value is slightly too large.

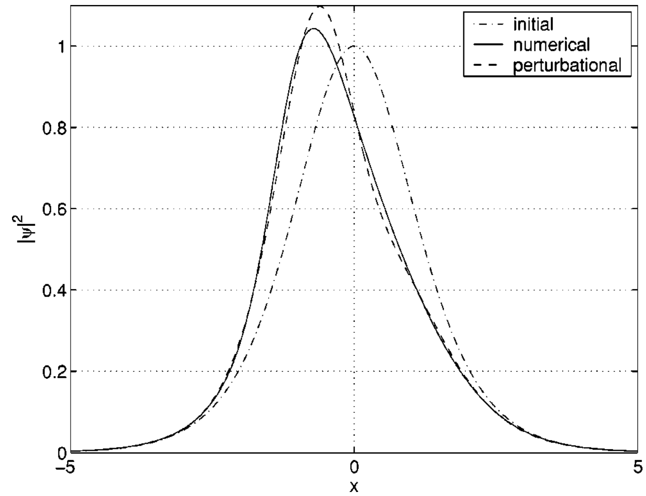


FIG. 4. Comparison of the asymmetric shapes predicted by analysis and numerical simulations, respectively.

## V. WAVE BREAKING

The nonlocal potential term, proportional to $\mu$ in Eq. (15), gives rise to a force, $F(x)=-\mu|\psi|^2$. This implies that the central parts of the bunch are affected by a stronger force than the wings, and will accelerate/decelerate more. In fact, this is the basic mechanism behind the steepening and the deformation of the bunch. It is clear that after a certain distance of propagation, the high-amplitude parts should overtake/be overtaken by the low-amplitude parts of the bunch. However, the finite dispersion will prohibit the development of an infinite amplitude gradient, and the "overtaking" between different parts of the bunch instead leads to the appearance of oscillations on the amplitude at the base of the steepening side of the bunch.

It is appropriate at this point to emphasize that this wave-breaking phenomenon does not involve a true shock formation with infinite gradients as is the case of wave breaking of plasma waves. Instead, this feature is analogous to the wave breaking in nonlinear defocusing Kerr media, [18,19], with the difference that in the latter case, the corresponding force is an odd function, which implies that the wave remains stationary and that the wave-breaking phenomenon occurs symmetrically on both sides of the pulse.

In order to estimate the characteristic length scale of the wave-breaking phenomenon, the perturbation result for the amplitude, Eq. (24), can be used. Thus, the order of magnitude of the wave-breaking distance, $z_{wb}$, is estimated as the shortest propagation distance for which the amplitude has a zero. This is easily shown to occur at

$$z_{wb} = \sqrt{\frac{3}{4|\alpha|a\mu A_0^2}}. \tag{27}$$

It is interesting to note that this approach gives the same result as the one used in Ref. [19], which was based on the

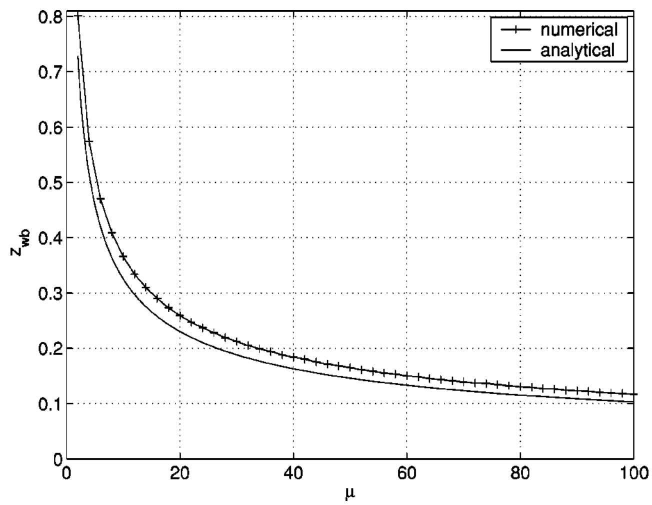


FIG. 5. The wave-breaking distance as predicted by analysis and numerical simulations, respectively.

local velocity shear in the pulse created by the nonlinearly induced chirp, provided the latter is generalized to include the mean acceleration of the pulse.

By increasing the propagation distance in Fig. 4, oscillations on the amplitude will start to occur at the base of the pulse on the steepening side, i.e., close to $x=-3$. Numerically we define the wave-breaking distance as the propagation distance where the amplitude acquires a second maximum. In Fig. 5, the analytical prediction for the wave-breaking distance is compared with the result of the numerical computations. The results show very good agreement, although the numerical results tend to be somewhat larger than predicted. However, since only an order-of-magnitude estimate has been made, the result is quite satisfactory. In particular, the analytic result predicts very well how the wave-breaking distance scales with $\mu$.

## VI. STATIONARY SOLUTIONS

As already discussed, the purely reactive case, corresponding to $\mu=0$, allows a soliton solution, Eq. (16), containing a finite number of particles. In order to investigate whether similar stationary solutions exist also in the general case, we return to Eqs. (20) and (21), which describe the evolution of the (real) amplitude and the phase of the wave, respectively. Based on our previous results, we will look for solutions that are stationary in an accelerated frame of reference, i.e., we introduce $s=x-\gamma z^2/2$, where the acceleration $\gamma$ now is unknown and has the character of an eigenvalue. Stationarity implies that the amplitude depends only on the coordinate $s$, i.e., $A=A(s)$. The phase variation can then be found explicitly, and the system becomes

$$\theta=\theta_0+C_1 z-\frac{\gamma z s}{2\alpha}-\frac{\gamma^2 z^3}{12\alpha}, \tag{28}$$

$$\alpha\frac{d^2A}{ds^2}+C_1A-\frac{\gamma s A}{2\alpha}+\kappa A^3+\mu A\int_{-\infty}^{s}A^2 ds'=0, \tag{29}$$

where $C_1$ is a constant, which acts as a second eigenvalue. However, by rescaling $\alpha$, $\gamma$, $\kappa$, and $\mu$, the equation can be rewritten as

$$\alpha\frac{d^2A}{ds^2}+A-\frac{\gamma s A}{2\alpha}+\kappa A^3+\mu A\int_{-\infty}^{s}A^2 ds'=0. \tag{30}$$

This normalization sets $C_1=1$, and the physical significance of this can be found by letting $\mu\to 0$ (implying also that $\gamma\to 0$). The fundamental soliton is then recovered from Eq. (29), and the phase is given by $C_1 z$. Thus, setting $C_1=1$ corresponds to normalizing with respect to the propagation constant.

Assume that a pulse-shaped stationary solution exists. This implies that $A\to 0$ as $s\to -\infty$, and asymptotically the field should satisfy the equation

$$\alpha\frac{d^2A}{ds^2}+A-\frac{\gamma s A}{2\alpha}=0, \tag{31}$$

which can be solved in terms of the Airy functions, $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, as

$$A=D_1\,\mathrm{Ai}\left(\frac{\gamma s-2\alpha}{\sqrt[3]{2\alpha^2\gamma^2}}\right)+D_2\,\mathrm{Bi}\left(\frac{\gamma s-2\alpha}{\sqrt[3]{2\alpha^2\gamma^2}}\right). \tag{32}$$

In order to obtain a solution containing a finite number of particles, it is necessary that $\gamma<0$ and that $D_2=0$. For pulse-like solutions, the amplitude of the solution must also vanish as $s\to\infty$ and the corresponding asymptotic equation is

$$\alpha\frac{d^2A}{ds^2}+(1+\mu W)A-\frac{\gamma s A}{2\alpha}=0. \tag{33}$$

Here, $W$ is the total number of particles, which has been assumed to be finite, and the corresponding solution is

$$\begin{aligned}A=&D_3\,\mathrm{Ai}\left(\frac{\gamma s-2(1+\mu W)\alpha}{\sqrt[3]{2\alpha^2\gamma^2}}\right)\\&+D_4\,\mathrm{Bi}\left(\frac{\gamma s-2(1+\mu W)\alpha}{\sqrt[3]{2\alpha^2\gamma^2}}\right).\end{aligned} \tag{34}$$

Since the acceleration has already been chosen to be negative, this implies that the asymptotic solution will be the sum of two oscillating Airy functions as $s\to\infty$. However, the total number of particles of such a solution is infinite, and a contradiction has been reached. Thus, we conclude that there are no stationary solutions to Eq. (14) containing a finite number of particles.

On the other hand, if the condition $A\to 0$ as $s\to\infty$ is relaxed, steplike solutions can be found. By assuming in Eq. (30) that $A\to A_\infty$ when $s\to\infty$, the integral term is asymptotically equal to $\mu A_\infty^3 s$. This term can cancel the term that gives rise to the Airy solutions, provided that

$$A_\infty=\sqrt{\frac{\gamma}{2\alpha\mu}}. \tag{35}$$

A solution of this type has been calculated numerically by using $\alpha=-1$, $\kappa=-1$, $\gamma=-1$, and $\mu=1$, and by choosing the amplitude for the asymptotic solution for negative $s$. The result has been plotted in Fig. 6, and it is seen that the pre-

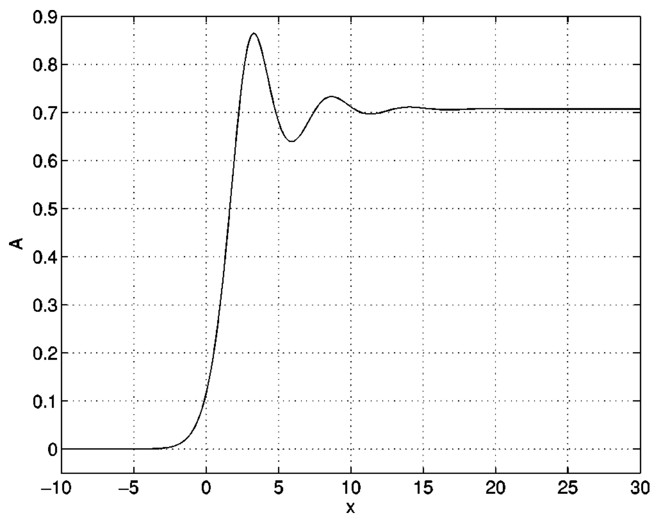


FIG. 6. A numerically obtained steplike solution.

dicted value for the asymptotic amplitude, $A_{\infty}=1/\sqrt{2}$, is correct.

## VII. CONCLUSIONS AND REMARKS

In conclusion, a generalized NLSE, describing the nonlinear longitudinal dynamics of high-energy charged particle beams in accelerators within the TWM approach, has been analyzed using both analytical and numerical methods. It has been discussed in qualitative physical terms how the inclusion of the resistive part of the coupling impedance gives rise to both an acceleration and a deformation of the particle bunch. These effects have been analyzed analytically using both a variational analysis and a direct perturbation analysis of the initial dynamics, and the results have been shown to be in good agreement with numerical simulations. It has also been shown that for impedances with a large resistive part, the deformation leads to self-steepening and eventually a wave-breaking phenomenon. The scale length for this effect has been estimated and has also been shown to be in good agreement with numerical results. Finally, it has been shown that no stationary pulselike solutions with a finite number of particles exist for the generalized NLS equation, but semi-infinite shock solutions are possible.

The results cannot be extrapolated to arbitrarily high amplitudes, since very steep gradients are predicted in that case and a more general equation, e.g., the full Eq. (2), must be used. It should, however, be noted that the asymmetric self-steepening does not lead to true shock formation in the sense of infinite derivatives. Instead, the steepening effect is balanced by the dispersive effects included in the model. We also emphasize that the wave-breaking phenomenon is not the analog of the wave breaking of plasma waves, where very steep gradients appear unless dispersion is included. Instead “wave breaking” refers to an optical phenomenon where the pulse steepens and starts to oscillate at its low intensity wings without any singular behavior. The maximum attainable slope is determined by the strength of the nonlinearity and consequently there is a maximum amplitude of the pulse at which the model breaks down. In a future work, the charged-particle beam dynamics will be investigated by taking into account the full Eq. (2).

[1] R. Fedele and G. Miele, Nuovo Cimento D **13**, 1527 (1991).
[2] R. Fedele, G. Miele, L. Palumbo, and V. G. Vaccaro, Phys. Lett. A **179**, 407 (1993).
[3] R. Fedele, F. Galluccio, and G. Miele, Phys. Lett. A **185**, 93 (1994).
[4] D. Anderson, R. Fedele, V. Vaccaro, M. Lisak, A. Berntson, and S. Johanson, Phys. Lett. A **258**, 244 (1999).
[5] R. Fedele and D. Anderson, J. Opt. B: Quantum Semiclassical Opt. **2**, 207 (2000).
[6] R. D. Ruth, in *An Overview of Collective Effects in Circular and Linear Accelerators*, Proceeding of a Topical Course of the Joint US-CERN School on Particle Accelerators, edited by M. Month and S. Turner (Springer-Verlag, Berlin, 1989).
[7] J. D. Lawson, *The Physics of Charged-Particle Beams* (Clarendon Press, Oxford, 1988).
[8] H. Schamel and R. Fedele, Phys. Plasmas **7**, 3421 (2000).
[9] L. Palumbo, V. G. Vaccaro, and M. Zobov, in *Wake Fields and Impedance*, Proceedings of CERN Accelerator School, Rhodos, Greece, 1993, edited by S. Turner (CERN, Geneva, 1995), Vol. I, p. 331.
[10] A. W. Chao, *Physics of Collective Beam Instability in High Energy Accelerators* (Wiley, New York, 1993).
[11] S. Koscielniak, S. Hancock, and M. Lindroos, Phys. Rev. ST Accel. Beams **4**, 044201 (2001).
[12] P. L. Colestock and L. K. Spentzouris, *The Future of Accelerator Physics*, edited by T. Tajima, AIP Conf. Proc. No. 356 (AIP, Woodbury, NY, 1996); P. L. Colestock *et al.* e-print, physics/9808035.
[13] L. K. Spentzouris, J.-F. Ostiguy, and P. L. Colestock, Phys. Rev. Lett. **76**, 620 (1996).
[14] H. Schamel, Phys. Rev. Lett. **79**, 2811 (1997).
[15] J.-M. Grießmeier, H. Schamel, and R. Fedele, Phys. Rev. ST Accel. Beams **5**, 024201 (2002).
[16] R. Fedele, S. De Nicola, V. G. Vaccaro, D. Anderson, and M. Lisak, in *Landau Damping in Nonlinear Schrödinger Equations*, Proceedings of the 18th Advanced ICFA Beam Dynamics Workshop on Quantum Aspect of Beam Physics, Capri, 2000, edited by P. Chen (World Scientific, Singapore, 2002), p. 483.
[17] D. Anderson, M. Lisak, and A. Berntson, Pramana **57**, 917 (2001).
[18] W. J. Tomlinson, R. H. Stolen, and C. V. Shank, J. Opt. Soc. Am. B **1**, 139 (1984).
[19] D. Anderson, M. Desaix, M. Lisak, and M. L. Quiroga-Teixeiro, J. Opt. Soc. Am. B **9**, 1358 (1992).